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Nonconvex Stochastic Optimization for Model Reduction

HAN-FU CHEN and HAI-TAO FANG
Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100080, P.R. China

Abstract. In this paper a global stochastic optimization algorithm, which is almost surely (a.s.) convergent, is applied to the model reduction problem. The proposed method is compared with the balanced truncation and Hankel norm approximation methods by examples in step responses and in approximation errors as well. Simulation shows that the proposed algorithm provides better results.

Key words: Stochastic optimization, Model reduction, Balanced truncation, Hankel norm approximation

1. Introduction

When designing control in order a system to possess some properties, a common approach is to work on its model. A real system may be described by a high order differential equation, which, however, may not be suitable for designers, because it is difficult, if not impossible, to design the optimal control for a high order system and to realize a sophisticated control law. Therefore, it is important to approximate a high order system by a lower order system without violating its stability. This is the topic of model reduction. The lower order approximate system, which control designers want to work with, is parameterized, and the parameters in the approximate system are selected to minimize the difference between the real system and the model, where the difference may be of their transfer functions, and expressed in H_2 or H_∞ norms. However, for a given set of parameters this difference may not exactly be observed because of the system uncertainties and random effects. In addition, in order to reduce the number of observations when minimizing the error norm, sometimes random directions [1] are used to form the finite differences of the minimized function. Therefore, this is a stochastic optimization problem.

This problem concerns with seeking for the extreme of a function $L(\cdot)$, defined on a multidimensional parameter space, on the basis of observations made on $L(\cdot)$ and possibly corrupted by noise. For solving the stochastic optimization problem one may apply stochastic approximation (SA) algorithms [2–5] which, however, may be stuck on a local extreme of $L(\cdot)$. As a rule, $L(\cdot)$ is nonconvex and the extremes of $L(\cdot)$ are not unique. This means that SA algorithms may generate a model which is not good enough. The simulated annealing method [6–8] may be applied for searching the global extreme of $L(\cdot)$, but it provides the convergence

of estimates to the global extreme only in probability and the convergence rate is rather slow. So, for modelling dealt with here it is of great importance to have an effective stochastic optimization algorithm that a.s. converges to the global minimizer of $L(\cdot)$.

In Section 2, the global stochastic optimization algorithm is described. In Section 3, the algorithm described in Section 2 is applied to model reduction. In Section 4, the balanced truncation and Hankel norm approximation, being the well-known methods for model reduction, are introduced with the purpose of comparison. In Section 5 the simulation results are presented comparing step responses and approximation errors.

2. Global stochastic optimization algorithms

We now describe an algorithm searching the global extreme of an unknown function $L(\cdot)$ which can be observed possibly with noise.

To be fixed, assume that the problem is to seek for the minimizer of $L(\cdot)$: $\mathbb{R}^l \to \mathbb{R}$. Let $(\Delta_k^j, j = 1, \ldots, l, k \in \mathbb{N})$ be a sequence of i.i.d. random variables such that

$$|\Delta_k^j| < a, \quad |\frac{1}{\Delta_k^j}| < b, \quad E(\frac{1}{\Delta_k^j}) = 0$$

$\forall i \in \{1, \ldots, l\}, \forall k \in \mathbb{N}$ where $a > 0$, $b > 0$. For example, one may take the Bernoulli's sequence of random variables, i.e., $P(\Delta_k^j = 1) = P(\Delta_k^j = -1) = \frac{1}{2}$.

Denote by x_k the estimate for the global minimizer x^0 at time k.

At time $k+1$ two observations are made:

$$y_{k+1}^+ = L(x_k + c_k\Delta_k) + \xi_{k+1}^+ \tag{1}$$

$$y_{k+1}^0 = L(x_k) + \xi_{k+1}^0, \tag{2}$$

where

$$\Delta_k = [\Delta_k^1, \ldots, \Delta_k^l]^\tau, \quad c_k > 0, \quad c_k \underset{k\to\infty}{\to} 0 \tag{3}$$

and ξ_{k+1}^+ and ξ_{k+1}^0 denote the noises. The following Kiefer-Wolfowitz (KW) algorithm with expanding truncations is used for local search [10,11]:

$$x_{k+1} = (x_k + a_k y_{k+1}) I_{\{\|x_k + a_k y_{k+1}\| \le M_{\sigma_k}\}} + x^* I_{\{\|x_k + a_k y_{k+1}\| > M_{\sigma_k}\}}, \tag{4}$$

$$\sigma_k = \sum_{j=0}^{k-1} I_{\{\|x_j + a_j y_{j+1}\| > M_{\sigma_j}\}}, \quad \sigma_0 = 0, \tag{5}$$

where $x^* \in \mathbb{R}^l$ is fixed, $M_k > 0$, $M_k \uparrow \infty$, $a_k > 0$, $a_k \to 0$, and $\sum_{k=0}^{\infty} a_k = \infty$. In (4), (5), y_{k+1} serving as the estimate for the gradient of $-L(\cdot)$ is defined by

$$\begin{aligned} y_{k+1} &= \frac{y_{k+1}^0 - y_{k+1}^+}{c_k}\Delta_k^{-1} \\ &= \frac{L(x_k) - L(x_k + c_k\Delta_k)}{c_k}\Delta_k^{-1} + \frac{\xi_{k+1}}{c_k}\Delta_k^{-1}, \end{aligned} \tag{6}$$

where

$$\Delta_k^{-1} \triangleq [\frac{1}{\Delta_k^1} \cdots \frac{1}{\Delta_k^l}]^\tau, \tag{7}$$

$$\xi_{k+1} = \xi_{k+1}^0 - \xi_{k+1}^+. \tag{8}$$

If the gradient $h(\cdot)$ of $L(\cdot)$ can be observed (possibly with noise), then in lieu of (1),(2) we may directly apply the observations

$$y_{k+1} = h(x_k) + \varepsilon_{k+1} \tag{9}$$

in (4) and (5), needless to introduce $\{\Delta_k\}$ and to form the difference (6).

Since the KW algorithm may be stuck on a local minimizer, we combine it with the random search (RS) [12] as follows.

Let the ith selection of RS for the global minimizer serve as the initial value of the ith run of the KW algorithm. In the ith run, the estimate and the noise are supplied with a superscript i and denoted by x_k^i and ξ_{k+1}^i, respectively, but for simplicity of notations, $\Delta_k^1, \ldots, \Delta_k^l$ will not be equipped with any additional superscript with following understanding that each Δ_k^i can only be used once.

Let RS be carried out according to a continuous probability density $p(\cdot)$. Let $f(i)$ and $e(i)$ be integer-valued increasing functions such that $f > e$ and $f(i+1) - f(i) \underset{i\to\infty}{\to} \infty$. Define $G(i) = f(i) - e(i)$. For example, we may take $f(i) = (a+i)^\alpha$, $a > 0$, $\alpha \geq 2$ and $e(i) = (a+i)^\beta$, $\alpha > \beta$.

Step 1. Local search from a randomly selected initial value.

Let $x_0^{(i)}$ be the outcome of the ith selection of RS according to the density $p(\cdot)$. Calculate $\{x_k^{(i)}\}$, $0 \leq k \leq G(i) - 1$ by (4)-(9) with initial value $x_0^{(i)}$, where

$$\begin{aligned} &a_k \triangleq a_k^{(i)} = a/(e(i)+k), a > 0, \\ &c_k = c/(e(i)+k)^\mu, c > 0, \end{aligned}$$

$\mu \in (\frac{1}{4}, \frac{1}{2})$, $x^* = x_0^{(i)}$ and $M_k = \|x_0^{(i)}\| \vee L_i \vee N_k$, where $\{L_i\}$ and $\{N_i\}$ are sequences of positive real numbers increasingly diverging to ∞ as $i \to \infty$. As an example, one may take $L_i = i$, or 2^i, and $N_i = i$, or 2^i.

At the end of this step, $x_{G(i)}^{(i)}$ is obtained.

Step 2. Estimating $L(x_{G(i)}^{(i)})$.

Corresponding to (2), the observation at $x_k^{(i)}$ is denoted by

$$y_{k+1}^{(i,0)} = L(x_k^{(i)}) + \xi_{k+1}^{(i,0)}, \tag{10}$$

and the estimate $\Lambda_{G(i)}^{(i)}$ for $L(x_{G(i)}^{(i)})$ is obtained by recursion

$$\Lambda_{k+1}^{(i)} = \Lambda_k^{(i)} + \frac{1}{k+1}(y_{k+2}^{(i,0)} - \Lambda_k^{(i)}),$$
$$\Lambda_0^{(i)} = 0, 0 \le k \le G(i) - 1. \tag{11}$$

Step 3. Comparison with previous results.

Let $\Lambda_{f(i)}$ be the previous estimate for the minimum of $L(\cdot)$. Take a sequence $\{\lambda(i)\}$ of real numbers such that $\lambda(i) > 0$ and $\lambda(i) \underset{i\to\infty}{\to} 0$ to serve as thresholds: Set $x_{f(i)} = x_{G(i)}^{(i)}$, if $\Lambda_{G(i)}^{(i)} + \lambda(i) < \Lambda_{f(i)} - \lambda(i)$. Otherwise, keep $x_{f(i)}$ unchanged, where $x_{f(i)}$ is the estimate for the global minimizer.

Step 4. Improving $x_{f(i)}$ by local search.

Improve $x_{f(i)}$ to $x_{f(i+1)}$ by use of (4) and (5) with $a_k = a/k$, $c_k = c/n^\mu$, $\mu \in (\frac{1}{4}, \frac{1}{2})$ and $k \ge f(i)$. Simultaneously, update the estimate Λ_k for $L(x_k)$ by

$$\Lambda_{k+1} = \Lambda_k + \frac{1}{n+1-f(i)}(y_{k+2}^0 - \Lambda_k), \tag{12}$$
$$\Lambda_{f(i)} = 0, \quad k \ge f(i). \tag{13}$$

By the end of this step $x_{f(i+1)}$ and $\Lambda_{f(i+1)}$ are derived. Then back to Step 1 for the $(i+1)$th selection of RS.

Steps 1–4 describe an optimization algorithm searching for global minimizer of $L(\cdot)$. This algorithm a.s. converges to the global minimizer of $L(\cdot)$ under quite general conditions. For example, the following conditions are sufficient.

(A1) $L(\cdot) : R^l \to R$ is such that ∇L is locally Lipschitz continuous, and there are constants k_0 and k_1 such that

$$k_0 \overset{\Delta}{=} \liminf_{\|x\|\to\infty} L(x) > L_{\min} (\overset{\Delta}{=} \min_{x\in R^l} L(x))$$

$$\|x^*\| < k, \text{ and } L(x^*) < \inf_{\|x\|=k_1} L(x),$$

and $\{L(x) : \nabla L(x) = 0, x \in \mathbb{R}^l\}$ is nowhere dense.

(A2) $\{\eta_k, \mathcal{F}_k^\eta\}$ and $\{\zeta_k, \mathcal{F}_k^\zeta\}$ are martingale difference sequences independent of $\{\Delta_k\}$ such that

$$\sup_k \{E(\eta_{k+1}^2 | \mathcal{F}_k^\eta) + E\eta_{k+1}^2\} < \infty,$$
$$\sup_k E\{(\zeta_{k+1}^2 | \mathcal{F}_k^\zeta) + E\zeta_{k+1}^2\} < \infty,$$

where $\{\eta_k\}$ and $\{\zeta_k\}$ denote the sequences obtained respectively from $\{\xi_k\}$ and $\{\xi_k^{(i,0)}\}$ ordered by their time of appearance in the algorithm.

(A3) $\liminf_{i\to\infty} e(i)i^{-(1+\delta)} > 0$ for some $\delta > 0$ and $f(i)/e(i) \to \infty$ as $i \to \infty$.

For the proof of a.s. convergence of the algorithm we refer to [11], where the noise conditions are weaker than (A2).

3. Application to model reduction

A real system may be modelled by a high order system which, however, may be too complicated for control design. In control engineering the order reduction for a model is of great importance. In the linear system case, this means that a high order transfer function $F(z)$ is to be approximated by a lower order transfer function. For this one may use methods like the balanced truncation (see, e.g., [13]) and the Hankel norm approximation (see, e.g., [13,14]). Both these methods are based on concept of the balanced realization. We are interested in recursively estimating the optimal coefficients of the reduced model by using the stochastic optimization algorithm presented in Section 2.

Let the high order transfer function $F(z)$ be

$$F(z) = \frac{\alpha_1 z^{n-1} + \alpha_2 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n}{z^n + \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + \beta_n} \tag{14}$$

and let it be approximated by a lower order transfer function $F_m(z) = C(z)/D(z)$. If $C(z)$ is of order $2s-1$ (or $2s$), then $D(z)$ is taken to be of order $2s$ (or $2s+1$). To be fixed, let us take $C(z)$ to be a polynomial of order $2s-1$ and $D(z)$ of order $2s$:

$$C(z) = c_1 z^{2s-1} + c_2 z^{2s-2} + \cdots + c_{2s-1} z + c_{2s}, \tag{15}$$

$$D(z) = (z^2 + d_{11} z + d_{21})(z^2 + d_{12} z + d_{22}) \cdots (z^2 + d_{1s} z + d_{2s}), \tag{16}$$

where coefficients c_i, $i = 1, \cdots, 2s$ should not be confused with stepsizes used in Steps 1–4. Write $F_m(z)$ as $F_m(c, d, z)$, where c and d stand for coefficients of $C(z)$ and $D(z)$

$$c = [c_1, c_2, \cdots, c_{2s-1}, c_{2s}]^\tau \text{ and } d = [d_{11}, d_{21}, \cdots, d_{1s}, d_{2s}]^\tau.$$

It is natural to take

$$\begin{aligned} L(c, d) &\triangleq \|F(z) - F_m(c, d, z)\|_2^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} |F(e^{j\omega}) - F_m(c, d, e^{j\omega})|^2 d\omega \end{aligned} \tag{17}$$

as the performance index of approximation. The parameters c and d are to be selected to minimize $L(c, d)$ under the constraint that $F_m(c, d, z)$ is stable. For simplicity of notations we denote

$$x = \begin{bmatrix} c \\ d \end{bmatrix},$$

and write $F_m(c, d, z)$ as $F_m(x, z)$.

Let us describe the x-set where $F_m(x, z)$ has the required property.

Stability requires that

$$|D(z)| \neq 0, \quad \forall z : |z| \geq 1.$$

This implies that

$$|d_{1i}| < 2, \quad i = 1, \ldots, s, \tag{18}$$

because d_{1i} is the sum of two complex-conjugate roots of $D(z)$.

If $d_{1i} > 0$, then $\frac{-d_{1i}-\sqrt{d_{1i}^2-4d_{2i}}}{2} > -1$, which yields $d_{1i} - 1 < d_{2i}$. If $d_{1i} < 0$, then $-d_{1i} - 1 < d_{2i}$, and hence

$$|d_{1i}| - 1 < d_{2i} < 1 \qquad i = 1, \ldots, s. \tag{19}$$

Set

$$D = \{d_{1i}, d_{2i} : |d_{1i}| < 2, |d_{1i}| - 1 < d_{2i} < 1, i = 1, \ldots, s\} \tag{20}$$

Identify $L(x)$, x, $\mathbb{R}^l$ and l appearing in Section 2 to $L(c, d)$, $\begin{bmatrix} c \\ d \end{bmatrix}$, $\mathbb{R}^{4s}$ and $4s$, respectively, for the present case.

We now apply the optimization algorithm given in Section 2 to minimizing $L(c, d)$ under constraint that the parameter x in $F_m(x, z)$ belongs to D. For this we first concretize Steps 1–4 described in Section 2.

Since $L(c, d)$ is convex in c for fixed d, we take the fixed initial value $c_0^{(i)} = (1, \cdots, 1)$ for any run i, and randomly select initial values only for d according to a distribution density $p(\cdot)$, which is defined as follows:

$$p(d) = \Pi_{i=1}^{s} p(d_{1i}, d_{2i}),$$

where $p(u, v) = q(v|u)q(u)$ with $q(u)$ and $q(v|u)$ being the uniform distributions over $[-2, 2]$ and $[|u| - 1, 1]$, respectively.

After $x_0^{(i)}$ having been selected in the ith run, the algorithm (4),(5) is calculated with

$$a_k \triangleq a_k^{(i)} = \frac{0.01}{e(i) + k + 1}, e(i) = (100 + i)^{1.5}.$$

As to observations, in stead of (6)–(8) we will use information about grandient because in the present case the gradient $h(c,d)(\triangleq h(x))$ of $L(c,d)$ can explicitly be expressed:

$$h(x) \triangleq h(c,d) \triangleq \nabla L(c,d) = \frac{1}{2\pi}\int_0^{2\pi} \nabla |F(e^{jw}) - F_m(c,d,e^{jw})|^2 dw$$
$$= -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}[(F(e^{jw}) - F_m(c,d,e^{jw}))\nabla\overline{F_m(c,d,e^{jw})}]dw. \qquad (21)$$

In the ith run the observation is denoted by $y_{k+1}^{(i)}$ and is given by

$$y_{k+1}^{(i)} = \frac{1}{100}\sum_{t=1}^{100} \mathrm{Re}[(F(e^{j(w_k+\frac{2\pi t}{100})}) - F_m(x_k^{(i)}, e^{j(w_k+\frac{2\pi t}{100})}))$$
$$\cdot \nabla\overline{F_m(x_k^{(i)}, e^{j(w_k+\frac{2\pi t}{100})})}],$$

where w_k is independently selected from $[0, 2\pi]$ by the uniform distribution, and $x_k^{(i)}$ stands for the estimate for $\begin{bmatrix} c \\ d \end{bmatrix}$ at time k in the ith run. It is clear that $y_{k+1}^{(i)}$ is an approximation to the integral (21) with $\begin{bmatrix} c \\ d \end{bmatrix} = x_k^{(i)}$. Therefore, we have observations in the form (9)

$$y_{k+1}^{(i)} = h(x_k^{(i)}) + \varepsilon_{k+1}^{(i)}.$$

The expanding truncation method used in (4) and (5) requires projecting the estimated value to a fixed point, if the estimated value appears outside an extending region. Let us denote it by Q_k. In (4) and (5) the spheres with extending radiuses M_{σ_k} serve as the extending regions Q_k, which are now modified as following.

Let us write

$$x_k^{(i)} = \begin{bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{bmatrix},$$

where $d_k^{(i)} \in D$. Define

$$D_k^{(i)} \triangleq \{|d_{1i}| \le 2(1 - \frac{1}{\tau_k^{(i)}}),$$
$$(1 - \frac{1}{\tau_k^{(i)}})(|d_{1i}| - 1 + \frac{1}{\tau_k^{(i)}}) \le d_{2i} \le 1 - \frac{1}{\tau_k^{(i)}}\} \subset D, \qquad (22)$$

$$Q_k^{(i)} \triangleq \mathbb{R}^{2s} \times D_k^{(i)}, \qquad (23)$$

where

$$\tau_k^{(i)} = \sum_{j=0}^{k-1} I_{\{(x_j^{(i)}+a_j^{(i)}y_{j+1}^{(i)}\in(Q_j^{(i)})^c\}}, \quad \tau_0^{(i)} = 1. \tag{24}$$

The expanding truncations in (4) and (5) are also modified:

$$\begin{aligned} x_{k+1}^{(i)} &= (x_k^{(i)} + a_k^{(i)} y_{k+1}^{(i))}) I_{[(x_k^{(i)}+a_k^{(i)}y_{k+1}^{(i)})\in Q_k^{(i)}]} \\ &\quad + (x_k^{(i)} + a_k^{(i)} y_{k+1}^{(i)})_p I_{[(x_k^{(i)}+a_k^{(i)}y_{k+1}^{(i)})\in(Q_k^{(i)})^c]} \end{aligned}$$

where $(x_k^{(i)} + a_k^{(i)} y_{k+1}^{(i)})_p$ means the projection of $x_k^{(i)} + a_k^{(i)} y_{k+1}^{(i)}$ to $Q_k^{(i)}$.

Take $f(i) = (100+i)^2$. Then after $G(i) = (100+i)^2 - (100+i)^{1.5}$ steps, $x_{G(i)}^{(i)}$ will be obtained.

Concerning steps 2–4, the only change consists in observations. We replace $y_{k+1}^{(i,0)}$ in steps 2–4 by $Y_{k+1}^{(i)}$ which is defined by

$$Y_{k+1}^{(i)} = \frac{1}{100} \sum_{t=1}^{100} \left| F(e^{j(w_k+\frac{2\pi t}{100})}) - F_m(x_k^{(i)}, e^{j(w_k+\frac{2\pi t}{100})}) \right|^2,$$

where w_k are independently selected from $[0, 2\pi]$ according to the uniform distribution for each k. Clearly, $Y_{k+1}^{(i)}$ is an approximation to $L(x_k^{(i)})(= L(c_k^{(i)}, d_k^{(i)}))$. Finally, take $\lambda(i)$ equal to $\frac{c}{\ln(i)}$.

Remark. We have just applied the stochastic approximation method to model reduction under the constraint that the approximate transfer function $F_m(c, d, z)$ is stable. It is clear that in lieu of requiring stability, other constraints, for instance, $\|c\| \le K_1$, $\|d\| \le K_2$ with $K_1 > 0$ and $K_2 > 0$ being constants, can be treated in a similar way. The only thing needed to do is to project the estimate to the boundary of the constrained region when it exits the region.

4. Model reduction methods used for comparison

In control theory there are several well-known model reduction methods such as model reduction by balanced truncation [13], Hankel norm approximation [13,14] among others. These methods depend on the balanced realization which is a state space realization method for a transfer matrix $F(s)$, keeping the Gramians for controllability and observability of the realized system balanced. To be precise, the linear system with constant coefficients

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is called a realization for a transfer function $F(s)$, if $F(s) = C(sI - A)^{-1}B + D$. If A is stable, then

$$P \triangleq \int_0^\infty e^{A\tau} B B^* e^{A^*\tau} d\tau \quad \text{and} \quad Q \triangleq \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} d\tau$$

are finite and called the controllability and observability Gramian respectively, where and hereafter X^* means taking complex conjugate and transpose for X. It can be shown that there is a similarity transformation T such that

$$PQ = T^{-1}\Lambda T,$$

where $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ is a diagonal matrix with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

Suppose the state x is transformed by T to $\hat{x} \triangleq Tx$. Then in the transformed coordinate system the controllability Gramian $\hat{P} = TPT^* = \Sigma$ equals the observability Gramian $\hat{Q} = (T^{-1})^* QT^{-1} = \Sigma$, where $\Sigma = diag(\sigma_1, \cdots, \sigma_n)$ and $\Sigma^2 = \Lambda$. It is clear that

$$\dot{\hat{x}} = TAT^{-1}\hat{x} + TBu, \quad y = CT^{-1}\hat{x} + Du$$

is also a realization of $F(s)$, which has equal Gramians for controllability and observability and is referred to a balanced realization. The ordered numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are called the Hankel singular values of the system, and σ_1 is called the Hankel norm of $F(s)$ and is denoted by

$$\|F(s)\|_H = \sigma_1.$$

For comparison with model reduction by stochastic optimization method described in Section 3, we will reduce the model by the following methods.

(i) Balanced truncation

Let (A, B, C, D) with $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = [C_1\ C_2]$ be a balanced realization of $F(s)$ with Gramian $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$, where $\Sigma_1 = \text{diag}(\sigma_1, \cdots \sigma_m)$ and $\Sigma_2 = \text{diag}(\sigma_{m+1}, \cdots, \sigma_n)$. Then the truncated system $F_m(s) \triangleq C_1(sI - A_{11})^{-1}B_1 + D$ is the model reduced by balanced truncation.

(ii) Hankel norm approximation

Let $F(s)$ be of McMillan degree n and let F_m be the set of transfer functions of McMillan degree equal to $m < n$. If

$$\inf_{F_m(s) \in F_m} ||F(s) - F_m(s)||_H = ||F(s) - \hat{F}_m(s)||_H,$$

then $\hat{F}_m(s)$ is the model reduced by minimizing the Hankel norm approximation.

These reductions are realized in the paper by using Matlab. For this, the discrete-time transfer functions $F(z)$ are transformed to the continuous time ones by using d2c provided in Matlab. Then the reduced systems are discretized to compute $L(c, d)$ for comparison.

5. Simulation results

As $F(z)$ we take a 10th order transfer function $F(z) = \alpha(z)/\beta(z)$ respectively for Example 1

$$\begin{aligned}\alpha(z) &= z^9 - 0.4z^8 + 0.08z^7 - 0.032z^6 + 0.0816z^5 - 0.0326z^4 \\ &\quad + 0.0288z^3 - 0.0115z^2 + 0.1296z - 0.0518, \\ \beta(z) &= z^{10} + 1.08z^8 + 0.8726z^6 + 0.6227z^4 + 0.4694z^2 + 0.1266.\end{aligned}$$

Example 2

$$\begin{aligned}\alpha(z) &= z^9 - 2.55z^8 + 4.62z^7 - 5.705z^6 + 6.1495z^5 - 5.9771z^4 \\ &\quad + 5.0659z^3 - 3.629z^2 + 1.7084z - 0.5523, \\ \beta(z) &= z^{10} - 3.55z^9 + 6.155z^8 - 5.688z^7 + 2.6317z^6 - 0.8835z^5 \\ &\quad + 2.5479z^4 - 4.714z^3 + 4.3881z^2 - 2.197z + 0.5194.\end{aligned}$$

Example 3

$$\begin{aligned}\alpha(z) &= z^9 + 1.1z^8 - 2.68z^7 - 2.858z^6 + 0.9821z^5 + 0.9453z^4 \\ &\quad - 0.1046z^3 - 0.828z^2 + 0.00858z + 0.002 \\ \beta(z) &= z^{10} - 3.6z^9 + 7.17z^8 - 10.836z^7 + 12.5713z^6 - 11.2381z^5 \\ &\quad + 7.9913z^4 - 4.3356z^3 + 1.5868z^2 - 0.3327z + 0.0296.\end{aligned}$$

Using the algorithm described in Section 3, for Examples 1–3 we obtain the approximate transfer functions of order 4, respectively denoted by $F1_4(z)$, $F2_4(z)$, and $F3_4(z)$ with

$$\begin{aligned}F1_4(z) &= \frac{0.9986z^3 + 0.0274z^2 - 0.7212z - 0.0865}{z^4 + 0.43z^3 + 0.4583z^2 + 0.1404z + 0.0757}, \\ F2_4(z) &= \frac{0.9435z^3 - 1.5672z^2 + 2.0739z - 1.4274}{z^4 - 2.7169z^3 + 3.3849z^2 - 2.1344z + 0.5807}, \\ F3_4(z) &= \frac{-2.9591z^3 + 8.2974z^2 + 3.5048z - 16.6678}{z^4 - 1.8622z^3 + 1.8829z^2 - 1.7667z + 0.7772}.\end{aligned}$$

Using Matlab we also derive the fourth order approximations for Examples 1–3 by balanced truncation and Hankel norm approximation, which are as follows:

$$F1_b(z) = \frac{1.0168z^3 + 0.3238z^2 - 0.2054z - 0.1490}{z^4 + 0.6979z^3 + 0.9651z^2 + 0.3961z + 0.3682},$$
$$F1_H(z) = \frac{1.4257z^3 + 1.4368z^2 - 0.2350z - 0.3201}{z^4 + 1.1498z^3 + 0.76z^2 + 0.4191z + 0.1022},$$
$$F2_b(z) = \frac{0.728z^3 - 0.7624z^2 + 0.9906z - 0.361}{z^4 - 2.6733z^3 + 3.5289z^2 - 2.3527z + 0.7628},$$
$$F2_H(z) = \frac{2.5851z^3 - 4.4881z^2 + 4.8781z - 1.6623}{z^4 - 2.4062z^3 + 2.9866z^2 - 1.8512z + 0.5752},$$
$$F3_b(z) = \frac{-6.6681z^3 + 9.5183z^2 + 2.8167z - 10.7083}{z^4 - 1.925z^3 + 1.9375z^2 - 1.8604z + 0.8718},$$
$$F3_H(z) = \frac{110.9644z^3 - 131.9689z^2 + 139.8014z - 135.0758}{z^4 - 1.2098z^3 + 1.2268z^2 - 1.1031z + 0.2512},$$

where the subscripts b and H denote the results obtained by balanced truncation and Hankel norm approximation respectively.

The approximation errors $L(c, d)$ are given in the following table, where by BT, HNA, and SO we mean balanced truncation, Hankel norm approximation, and stochastic optimization, respectively.

	BT	HNA	SO
Example 1	0.1694	0.3136	0.0641
Example 2	6.8254	7.3206	2.9976
Example 3	1349.9	14820	761.8623

From this table we see that the algorithm presented in Section 3 gives less approximation errors in H_2-norm in comparison with other methods.

We now compare approximation errors in H_∞-norm and compare step responses between the approximate models and the true one by Figure 1.

In the figures of step response

- the solid lines (——) denote the true high order systems;
- the dashed lines (- - -)denote the system reduced by Hankel norm approximation;
- the dotted lines ($\cdots\cdots$) denote the system reduced by balanced truncation;
- The dotted-dashed lines ($-\cdot-$) denote the systems reduced by the stochastic optimization method given in Section 3.

In the figures of the approximation error $|F(e^{jw}) - F_m(c, d, e^{jw})|^2$, $w \in [0, 2\pi]$

- the solid lines (——) denote the systems reduced by the stochastic optimization method;
- the dashed lines (- - -) denote the system reduced by Hankel norm approximation;
- the dotted lines (· · · · ·) denote the system reduced by balanced truncation.

This figure shows that the algorithm given in Section 3 gives less approximation error in H_∞-norm in comparison with other methods for Example 1 and the intermediate error in H_∞-norm for Examples 2 and 3. Concerning step responses, the algorithm given in Section 3 provides better approximation in comparison with other methods for all three examples.

6. Conclusion

The problem of order reduction for linear systems in essence is a nonconvex optimization problem. Because of nonconvexity of the minimized function, being the approximation error as a function of coefficients of the reduced model, a convex optimization algorithm may lead to an unsatisfactory model that greatly differs from the real system. The widely accepted model reduction methods in control theory such as balanced truncation and Hankel norm approximation provide satisfactory results in many cases, but the approximation accuracy is not optimized. In this paper the global stochastic optimization algorithm proposed by the authors is applied to reducing the system orders.

The simulation shows that the stochastic optimization method works very well: it provides the step response as well as the transfer function of the reduced system closer to those of the true system, in comparison with other methods. Therefore, the proposed stochastic optimization algorithm is promising to be successfully used in model reduction. As a matter of fact, it may be applied to other problems in control where parameter optimization is concerned, for example, the problem of optimizing weights in neural networks. Of course, for different problems, the specific feature of the optimized function in question should be taken into account. This would lighten the computational burden and improve accuracy. For example, in Section 3 $L(c, d)$ is nonconvex in $\begin{bmatrix} c \\ d \end{bmatrix}$, but it is convex in c for fixed d. So the search for initial value is carried out only for d, and this greatly saves the computational time. For the nonconvex stochastic optimization algorithm itself, new combinations of global search with local optimization other than the one presented in the paper may be considered for further research.

7. Acknowledgements

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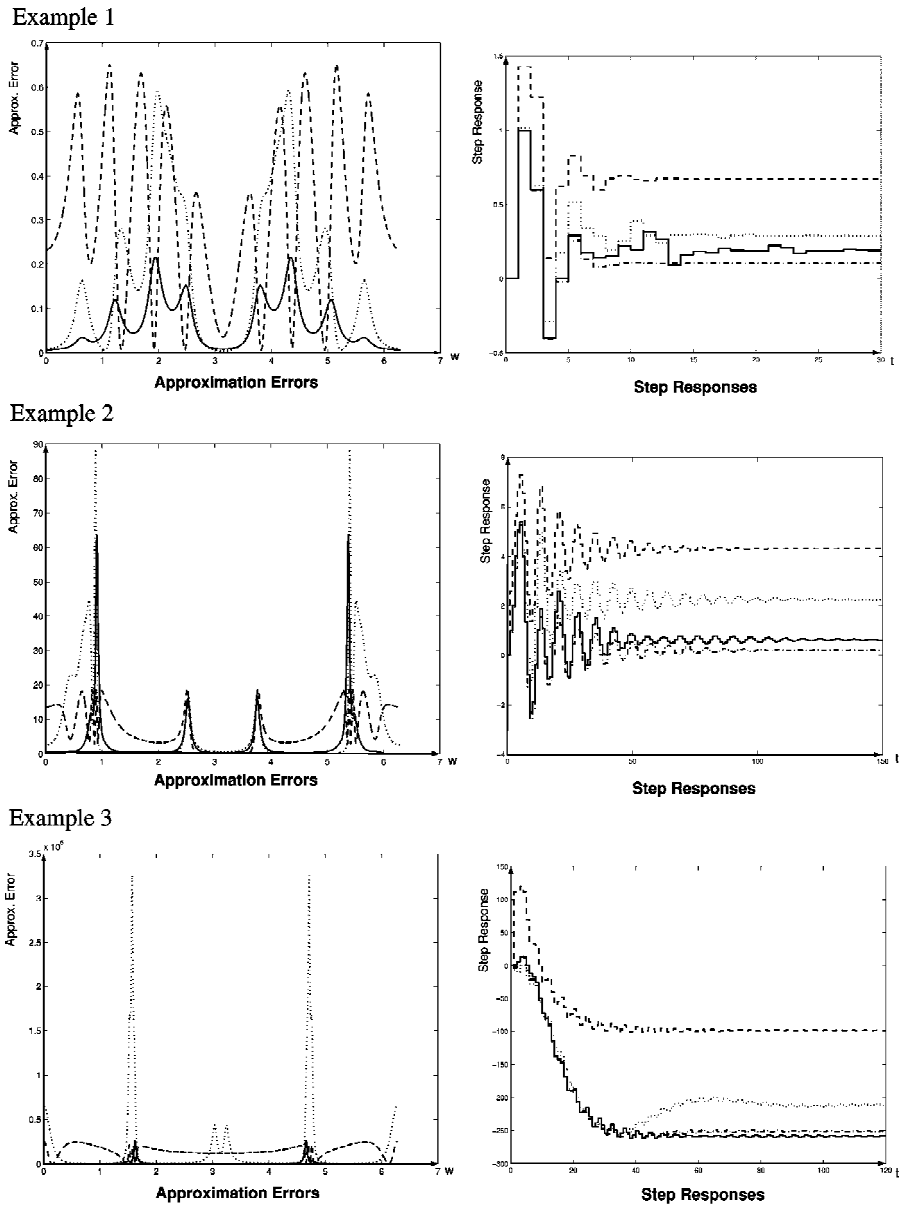

Figure 1.

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